A History of Exchange Option Pricing Models

Gerald H L Cheang*

UniSA STEM, Centre for Industrial and Applied Mathematics

University of South Australia

University of South Australia

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*Main Collaborator: Len Patrick Dominic M Garces; Other Collaborators: Carl Chiarella, Guanghua Lian

1 Overview of Talk

- What are options?
- History of Modern Option Pricing Theory (on a single stock/asset) from around 1950.
- The Black-Scholes-Merton (1973) stock pricing model.
- The Black-Scholes call (and put) option pricing formulas on a single stock.
- The Margrabe (1978) Exchange Option (between two different stocks) pricing formula.
- Inadequacy of the Black-Scholes-Merton Geometric Brownian Motion stock pricing model.

- Other extensions to the Black-Scholes-Merton model including adding jump-diffusion terms and/or stochastic volatility.
- Cheang and Chiarella (2011) extended the Margrabe formula to two stocks that also exhibit jumps. They also considered American style versions of such exchange options.
- Cheang and Lian (2015) considered Perpetual Exchange Options under jump-diffusion dynamics.
- Cheang and Garces (2020), Garces and Cheang (2021) considered both European and American style Exchange Options under combined stochastic volatility and jump-diffusion models.

2 What are Options?

- Since the Middle Ages in Europe (starting in Italy), there were already versions of transactional contracts for commodities quite similar to some of the modern day financial instruments such as futures, forwards and call/put options.

- Similarly in Japan, farmers and rice traders have also entered into similar contracts for the hedging of the price of rice over the centuries.

- In modern times, we have financial instruments such as futures contract, call and put options, exchange options and spread options.

- In Bursa Malaysia, call and put options on palm oil, as well as futures on tin are traded.

- A futures contract is binding on both seller and buyer of the underlying asset. A price is agreed when the contract is struck. At the maturity time T, the buyer will buy the agreed amount of the asset at the agreed price and the seller has to sell that amount at that price. Since the contract is binding, there isn't an initial price to pay to enter the contract. The payoff from the buyer's point of view is $S_T - K$ and from the seller's is $K - S_T$.

- A call option allows the holder of the option to buy 1 unit of the underlying asset at the agreed strike price K. The holder of the option will exercise his call option to buy the asset if the actual price of the asset at the time of exercise is higher than the strike price. The seller of the option has to honour the option and deliver 1 unit of the asset at the lower price K.

- A put option allows the holder of the option to sell 1 unit of the underlying asset at the agreed strike price K. The holder of the option will exercise his put option to sell the asset if the actual price of the asset at the time of exercise is lower than the strike price. The seller of the option has to honour the option and buy 1 unit of the asset at the higher price K.
- There are European and American versions of the call/put options. For the European version, the option can only be exercised at the maturity time T. For the American version, the option can be exercised any time till maturity time T.
- Since the holder of the option can choose not to exercise the option if conditions are not favourable to them, a fair price is needed to paid out to the seller of the option at time $t = 0$. This option pricing formula will depend on the pricing model of the underlying asset.

- For the European option, the payoff (or gain) at maturity of the holder of the option is $(S_T - K)^+ = \max(S_T - K, 0)$ for the call option and $(K - S_T)^+ = \max(K - S_T, 0)$ for the put option.
- For the American option when exercised at τ, where $0 \leq \tau \leq T$, the payoff (or gain) at the exercise time of the holder of the option is $(S_\tau - K)^+ = \max(S_\tau - K, 0)$ for the call option and $(K - S_\tau)^+ = \max(K - S_\tau, 0)$ for the put option.
- An Exchange option allows the holder of the option to exchange one stock/asset for another and is commonly used in foreign exchange markets, bond markets and stock markets.
- Margrabe (1978) derived a pricing formula for the European style exchange option in the Black-Scholes framework where the stock prices are modelled by correlated Geometric Brownian Motion processes.

- For the Exchange option when 1 unit of asset 2 is exchanged for 1 unit of asset 1, the payoff is $(S_{1,T} - S_{2,T})^+ = \max(S_{1,T} - S_{2,T}, 0)$ for the European version when exercised at maturity time T, and the payoff is $(S_{1,\tau} - S_{2,\tau})^+ = \max(S_{1,\tau} - S_{2,\tau}, 0)$ if exercised at time $t = \tau$, where $0 \leq \tau \leq T$, for the American version.

- In this presentation, I will also highlight my work with my collaborators in extending Margrabe's (1978) result under different scenarios for the underlying assets such as with jump-diffusion and combined stochastic volatility jump-diffusion.

3 The Black-Scholes-Merton (1973) Model

- A much earlier model by Bachelier (1900) in his doctoral thesis assumed that the underlying asset price S_t was Normally distributed.
- It was not until the mid 1950s onwards that alternative models postulated that it was the log returns $\ln \frac{S_t}{S_0}$ that is normally distributed.
- In the Black-Scholes (1973) and Merton (1973) model, the stock price dynamics is driven by the stochastic differential equation

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t, \tag{1}$$

  where μ is the drift of the stock, σ is the volatility and W_t is 1-dimensional standard Brownian motion.
- The initial models assumed constant drift and volatility.

- We can interpret $\frac{dS_t}{S_t}$ as the relative instantaneous change in the stock price.
- A solution to Equation (1) using Itô's Lemma yields

$$S_t = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right). \tag{2}$$

- However stocks, options and other assets that are traded in a frictionless market are only fairly priced under a risk-neutral setting when all the market players (i.e. buyers and sellers) are "risk-neutral".
- A "risk-neutral" investor is indifferent to any asset.

- Under the risk-neutral setting (with respect to a risk-neutral probability measure $\mathbb{Q}$), the average price of all assets increase exponential by the risk-free interest rate r, thus under $\mathbb{Q}$, the stock price is now driven by

$$\frac{dS_t}{S_t} = rdt + \sigma d\tilde{W}_t, \tag{3}$$

where $\tilde{W}_t$ is standard Brownian Motion under $\mathbb{Q}$.

- The solution to Equation (3) is

$$S_t = S_0 \exp\left(\left(r - \frac{1}{2}\sigma^2\right)t + \sigma\tilde{W}_t\right). \tag{4}$$

- For a European style option where the payoff at maturity time T is $g(S_T)$, a fair price to pay for the option at time $t = 0$ is $\mathbb{E}_{\mathbb{Q}}[e^{-rT}g(S_T)]$, where r is the risk-free rate. Hence for the European call option, it is $\mathbb{E}_{\mathbb{Q}}[e^{-rT}(S_T - K)^+]$ and for the European put option it is $\mathbb{E}_{\mathbb{Q}}[e^{-rT}(K - S_T)^+]$.

- Black, Scholes and Merton also applied hedging strategies for a no-arbitrage, self-financing portfolio of the underlying stock and risk-free bond/cash to replicate the value of the option from time $t = 0$ to maturity and came up with a partial differential equation known as the Black-Scholes PDE, which is,

$$\partial_t C_t(S_t) + rS_t\partial_s C_t(S_t) + \frac{1}{2}\sigma^2 S_t^2 \partial_{ss} C_t(S_t) = rC_t(S_t), \qquad (5)$$

where $C_T(S_T) = g(S_T)$ is the terminal boundary condition.

- For the call option option, the pricing formula for $0 \leq t \leq T$ is

$$C_t(S_t) = S_t \Phi(d_1) - K e^{-r(T-t)} \Phi(d_2), \tag{6}$$

where $d_1 = \frac{\ln \frac{S_t}{K} + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$, $d_2 = d_1 - \sigma\sqrt{T-t}$ and $\Phi(\ )$ is the standard Normal cumulative distribution function.

- In Equation (6), we also see that at any time t, the value of the call option $C_t(S_t)$ can be replicated with a self-financing portfolio of $\Phi(d_1)$ units of stock and borrowing $K e^{-r(T-t)} \Phi(d_2)$ of cash.

- It is possible to extend the Black-Scholes-Merton model to the case where the underlying stock pay a continuous dividend yield.

- For an American style option, the price of an American option is

$$C_t^A(S_t) = C_t^E(S_t) + C_t^P(S_t), \tag{7}$$

where $C_t^E(S_t)$ is the European price and $C_t^P(S_t)$ is the early exercise premium.

- In the absence of any dividends, it turns out that the optimal exercise time for the call option is at maturity and hence both the American call and the European call must have the same price.
- With dividends, the optimal time to exercise an American call option could be before maturity.
- For the put option, there are circumstances where it is optimal to exercise early, thus the price of the American put must be higher than the European put and there is an early exercise premium.

- There are no close-form expressions for the American put option price since this involves solving for a free boundary problem involving the PDE.

- From an optimal stopping problem point of view, the price of an American style option at time $t = 0$ is

$$C_0^A(S_0) = \sup_{0 \le \tau \le T} \mathbb{E}_{\mathbb{Q}}[e^{-r\tau} g(S_\tau)], \tag{8}$$

  where τ is the exercise time and $g(\ )$ is the payoff function of the option.

- Perpetual options do not mature and can be exercised at any time, thus

$$C_0^{\mathrm{Per}}(S_0) = \sup_{\tau \ge 0} \mathbb{E}_{\mathbb{Q}}[e^{-r\tau} g(S_\tau)]. \tag{9}$$

4 Margrabe's (1978) Exchange Option Model

- Margrabe (1978) extended the Black-Scholes-Merton option pricing model to the exchange option, while still retaining the Geometric Brownian Motion model for the two stocks.
- The exchange option can be considered a special case of a multi-asset spread option (terminal payoff $(\sum_{i=1}^{n} a_i S_{i,T} - K)^+$).
- Outperformance bonuses for company board members can be considered as a form of the exchange option
- The terminal payoff condition for the exchange option is $(S_{1,T} - S_{2,T})^+$ if 1 unit of stock 2 were to be exchanged for 1 unit of stock 1.

- The stock prices dynamics for the Margrabe model are

$$\frac{dS_{i,t}}{S_{i,t}} = \mu_i dt + \sigma_i dW_{i,t}, \tag{10}$$

for $i = 1, 2$ where $(W_{1,t}, W_{2,t})$ is 2-dimensional correlated Brownian Motion with $dW_{1,t} \, . \, dW_{2,t} = \rho dt$.

- Using Itô's Lemma again yields

$$S_{i,t} = S_{i,0} \exp\left(\left(\mu_i - \frac{1}{2}\sigma_i^2\right) t + \sigma_i W_{i,t}\right), \tag{11}$$

for $i = 1, 2$.

- Since the terminal payoff condition is $(S_{1,T} - S_{2,T})^+$, the price of the exchange option at time zero is $\mathbb{E}_{\mathbb{Q}}[e^{-rT}(S_{1,T} - S_{2,T})^+]$.

- The pricing formula for the exchange option over $0 \le t \le T$ is

$$C_t(S_{1,t}, S_{2,t}) = S_{1,t}\Phi(d_1) - S_{2,t}\Phi(d_2), \tag{12}$$

where $d_1 = \frac{\ln \frac{S_{1,t}}{S_{2,t}} + \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}$, $d_2 = d_1 - \sigma\sqrt{T-t}$, $\Phi(\,)$ is the standard Normal cumulative distribution function, and
$\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$.

- The exchange option price in Equation (12) is also a solution to the PDE

$$\begin{aligned}\partial_t C_t(S_{1,t}, S_{2,t}) + \frac{\sigma_1^2}{2}\partial_{11}C_t(S_{1,t}, S_{2,t}) + \frac{\sigma_2^2}{2}\partial_{22}C_t(S_{1,t}, S_{2,t}) \\ + \rho\sigma_1\sigma_2\partial_{12}C_t(S_{1,t}, S_{2,t}) = 0,\end{aligned} \tag{13}$$

with boundary condition $C_T(S_{1,T}, S_{2,T}) = (S_{1,T} - S_{2,T})^+$.

- In Equation (12), we see that to replicate the value of the exchange option $C_t(S_{1,t}, S_{2,t})$, we need $\Phi(d_1)$ units of stock 1 and short-sell $\Phi(d_2)$ units of stock 2.
- If there are no dividends for both stocks, it turns out that it is optimal to exercise the American exchange option at maturity, hence it has the same value as the European exchange option.
- If one or both stocks are dividend paying, then it may be optimal to exercise the American exchange option before maturity, depending on the dividends for each stock.

5 Limitations of the Black-Scholes-Merton Model

- Since Brownian Motion paths are continuous almost-everywhere in time (even though non-differentiable almost-everywhere), Geometric Brownian Motion paths also exhibit the same property.

- Thus stock price paths modelled by the Black-Scholes-Merton pure diffusion model also exhibit the same properties.

- The Black-Scholes-Merton model also assumes constant volatility and constant risk free rate.

- Under the Black-Scholes-Merton model, the stock price is log-normally distributed.

- In practice, stock price paths exhibit jumps when there are idiosyncratic shocks or systemic shocks.
- A phenomenon known as the volatility smile is also observed, when plotting the implied volatility of an option versus the strike price, which implies that volatility is not constant.
- If volatility were constant, the implied volatility plot versus the strike price would be a flat line.
- Empirical studies have shown that the classical Black-Scholes assumption that asset prices are log-normally distributed is insufficient to capture pertinent features of asset returns such as heavy tails, volatility clustering, and implied volatility smiles and skews (Cont 2001, Cont and Tankov 2004, Kou 2008).

- Due to the limitations, alternative models have been proposed such as jump-diffusion models (Merton 1976, Naik and Lee 1990, Pham 1997, Kou 2002), stochastic volatility models (Hull and White 1987, Stein and Stein 1991, Heston 1993), and combinations of stochastic volatility and jump-diffusion models (Bates 1996, Bakshi et al. 1997, Scott 1997).

- These alternative models were mainly used to derive the prices of call and put options on a single stock/asset.

- For example, in the Merton (1976) model, the underlying stock price for the call option is

$$S_t = S_0 \exp\left(\left(\mu - \lambda\kappa - \frac{1}{2}\sigma^2\right)t + \sigma W_t + \sum_{n=1}^{N_t} Y_i\right), \tag{14}$$

where N_t is a homogeneous Poisson process with arrival rate λ that models the arrival of the jumps, Y_i are independently and identically distributed as Normal $N(\alpha, \delta^2)$, and $\kappa = \mathbb{E}_{\mathbb{P}}[e^Y - 1]$ is the expected relative increment due to the jumps.

- Merton derived a pricing formula for a European call option on a stock driven by the jump-diffusion model in Equation (14).

- In the Heston (1993) model, the stock price and volatility dynamics are given by

$$\frac{dS_t}{S_t} = \mu dt + \sqrt{v_t} dW_{1,t}, \tag{15}$$

$$dv_t = \theta(\alpha - v_t)dt + \sigma_v \sqrt{v_t} dW_{2,t}, \tag{16}$$

where $dW_{1,t} \, . \, dW_{2,t} = \rho dt$.

- The stock price in this model is

$$S_t = S_0 \exp\left(\int_0^t \left(\mu - \frac{1}{2} v_u\right) du + \int_0^t \sqrt{v_u} dW_{1,u}\right). \tag{17}$$

- Heston derived a close-form formula in terms of the Fourier transform of the call option price.

6 Exchange Option under Jump-Diffusion

- While there has been numerous and various extensions of the Black-Scholes-Merton setting for pricing options written on a single stock, there was little work done on extending the setting for options written on multiple stocks under jump-diffusion models prior to the mid-2000s.
- Cheang et al. (2006) and Cheang and Chiarella (2011) extend Margrabe's (1978) exchange option model to the jump-diffusion framework.
- In Cheang and Chiarella (2011), the two stock prices $i = 1, 2$ with continuously paying dividend rate ξ_i under an equivalent martingale measure $\mathbb{Q}$ are given by

$$S_{i,t} = S_{i,0} \exp\left[\left(r - \xi_i - \frac{\sigma_i^2}{2} - \tilde{\lambda}\tilde{\kappa}_i - \tilde{\lambda}_i\tilde{\kappa}_{Z_i}\right)t + \sigma_i \widetilde{W}_{i,t} + \sum_{m=1}^{N_{i,t}} Z_{i,m} + \sum_{n=1}^{N_t} Y_{i,n}\right]. \tag{18}$$

- In Equation (18), $(\widetilde{W}_{1,t}, \widetilde{W}_{2,t})$ are correlated standard Brownian motion components where $d\widetilde{W}_{1,t} \ . \ d\widetilde{W}_{2,t} = \rho dt$; N_t is a Poisson process with rate $\tilde{\lambda}$ for the common jumps to both stocks, the jump-sizes $\boldsymbol{Y}_n$ are independently and identically distributed as multivariate normal $MVN(\tilde{\boldsymbol{\alpha}}, \Sigma_{\boldsymbol{Y}})$, where $\tilde{\boldsymbol{\alpha}} = (\tilde{\alpha}_1, \tilde{\alpha}_2)^\top$ and $\Sigma_{\boldsymbol{Y}} = \begin{pmatrix} \delta_1^2 & \rho_{\boldsymbol{Y}}\delta_1\delta_2 \\ \rho_{\boldsymbol{Y}}\delta_1\delta_2 & \delta_2^2 \end{pmatrix}$; the idiosyncratic jump arrivals for each stock $N_{i,t}$ are Poisson processes with arrival rate $\tilde{\lambda}_i$ and the idiosyncratic jump-sizes $Z_{1,k}$ and $Z_{2,m}$ are respectively independently and identically distributed as Normal $N(\tilde{\alpha}_{ii}, \delta_{ii}^2)$ under $\mathbb{Q}$.
- $\tilde{\kappa}_i = \mathbb{E}_{\mathbb{Q}}[e^{Y_i} - 1]$ and $\tilde{\kappa}_{Z_i} = \mathbb{E}_{\mathbb{Q}}[e^{Z_i} - 1]$ are the various expected relative jump-size increments in Equation (18).

- A jump-diffusion market is incomplete (see e.g. Cont and Tankov (2004)) and there are many equivalent martingale measures $\mathbb{Q}$, so in what follows, we assume that we have already chosen a particular equivalent martingale measure $\mathbb{Q}$.
- In the literature, there are various ways for which an equivalent martingale measure can be chosen in an incomplete market.
- In Cheang and Chiarella (2011), a close-form formula for the European exchange option under the setting of Equation (18) is derived.

- Although no close-form formula for the American exchange option exists, they are also able to show that

$$C_t^A(S_{1,t}, S_{2,t}) = C_t^E(S_{1,t}, S_{2,t}) + C_t^P(S_{1,t}, S_{2,t}), \tag{19}$$

where $C_t^E(S_{1,t}, S_{2,t})$ is the European price and $C_t^P(S_{1,t}, S_{2,t})$ is the early exercise premium.

- Since there are dividends in the model given by Equation (18), there will be occasions when it is optimal to exercise the American option before maturity.

- Interpretation of the early exercise premium term $C_t^P(S_{1,t}, S_{2,t})$ when jumps happen to occur at the instant of time of exercise is also given in Cheang and Chiarella (2011).
- The early exercise boundary for the American exchange option at time t where $0 \leq t \leq T$ is shown in Figure 1, when $S_{1,t} = s_1$ and $S_{2,t} = s_2$. It is time dependent.

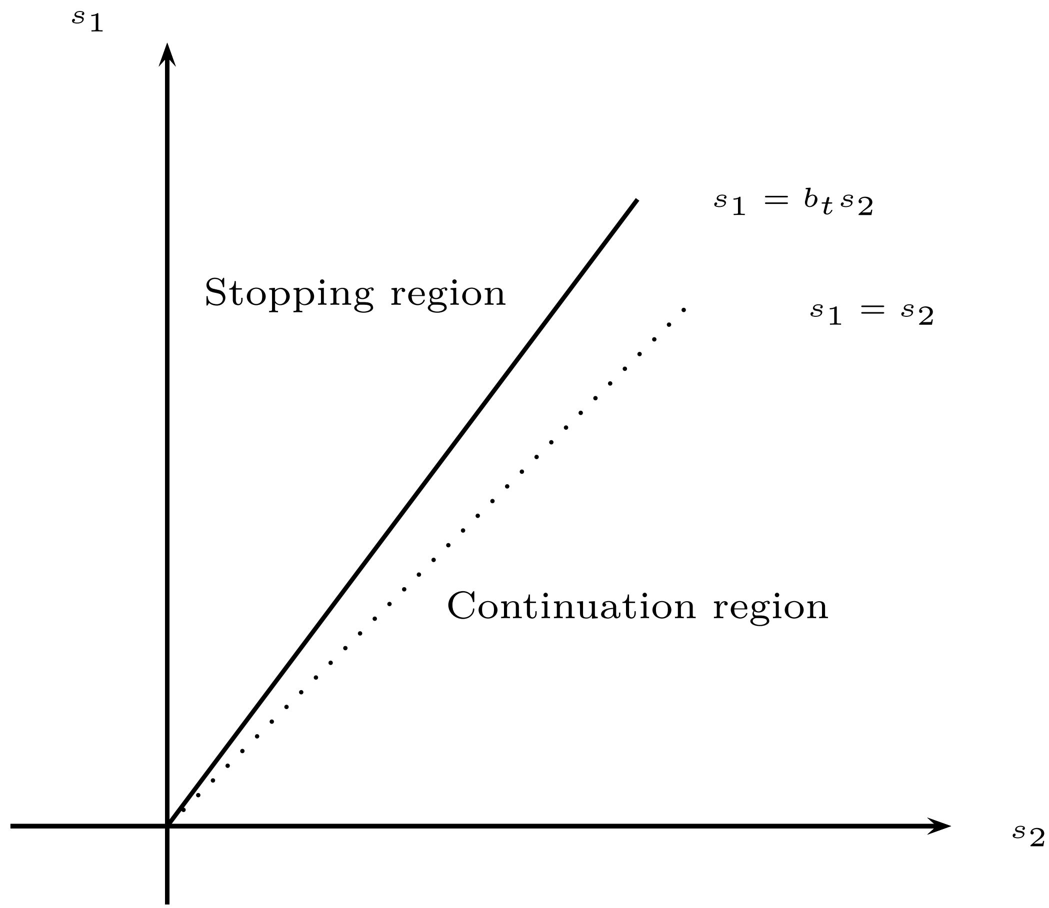

Figure 1: Continuation and stopping regions for the American exchange option, at a given time t. The early exercise boundary is the line $s_1 = b_t s_2$.

7 Perpetual Exchange Options

- The perpetual (American) exchange option under pure-diffusion dynamics was first considered by Gerber and Shiu (1996a and 1996b) who employed optimal stopping techniques, and was later generalized by Wong (2008).
- Liu and Liu (2009) consider the pricing of the perpetual option, again under pure-diffusion dynamics, from the perspective of solving a two variable free boundary problem.
- The perpetual call and put options under pure-diffusion dynamics have been priced by Gerber and Shiu (1994), whereas the pricing of perpetual options on single stocks under jump-diffusion dynamics is found in Gerber and Shiu (1994, 1998).

- Cheang and Lian (2015) extended the perpetual option results of Gerber and Shiu (1996a and 1996b) for underlying stocks driven by jump-diffusion dynamics.

- The stock price dynamics are given by

$$\frac{dS_{i,t}}{S_{i,t-}} = (r - \xi_i - \lambda_i \kappa_i)dt + \sigma_i dW_{i,t} + (e^{Y_i} - 1)dN_{i,t}, \qquad (20)$$

  for $i = 1, 2$ under a prescribed equivalent martingale measure $\mathbb{Q}$.

- $W_{1,t}$ and $W_{2,t}$ are components of a bivariate correlated Brownian motion process, where $dW_{1,t}\ dW_{2,t} = \rho dt$, and ρ is the instantaneous correlation between the two Brownian motion components and $|\rho| < 1$.

- The $N_{i,t}$ are independent Poisson processes with arrival rate λ_i and $\kappa_i = \mathbb{E}_{\mathbb{Q}}[e^{Y_i} - 1]$ is the expected relative jump-size increment for the ith stock.

- As with the pure-diffusion counterpart, this is an optimal stopping problem where

$$C_t^{\text{Per}}(S_{1,t}, S_{2,t}) = \sup_{\tau \geq t} e^{-r(\tau - t)} \mathbb{E}_{\mathbb{Q}}[(S_{1,\tau} - S_{2,\tau})^+ | \mathcal{F}_t]. \tag{21}$$

- From the known properties of the perpetual call and put options under pure diffusion as well as jump-diffusion dynamics. (Gerber and Shiu (1996a, 1996b)) and that of a perpetual option on a single asset under jump-diffusion dynamics (Gerber and Shiu (1994, 1998)), we can assume that our perpetual exchange option price is independent of time, thus

$$C_t^{\text{Per}}(S_{1,t}, S_{2,t}) = C^{\text{Per}}(S_{1,t}, S_{2,t}). \tag{22}$$

- Analogous to the case of the American exchange option under jump-diffusion dynamics in Cheang and Chiarella (2011), there is an early exercise boundary that separates the early exercise region and continuation region.
- The key difference is that the early exercise boundary is independent of time.
- The early exercise boundary b separates the early exercise region $\{(S_{1,t}, S_{2,t}) : S_{1,t} \geq bS_{2,t}\}$ from the continuation region $\{(S_{1,t}, S_{2,t}) : S_{1,t} < bS_{2,t}\}$ for $b > 1$.
- The early exercise boundary for the perpetual exchange option at time t where $0 \leq t \leq T$ is shown in Figure 2, when $S_{1,t} = s_1$ and $S_{2,t} = s_2$. It is time independent.

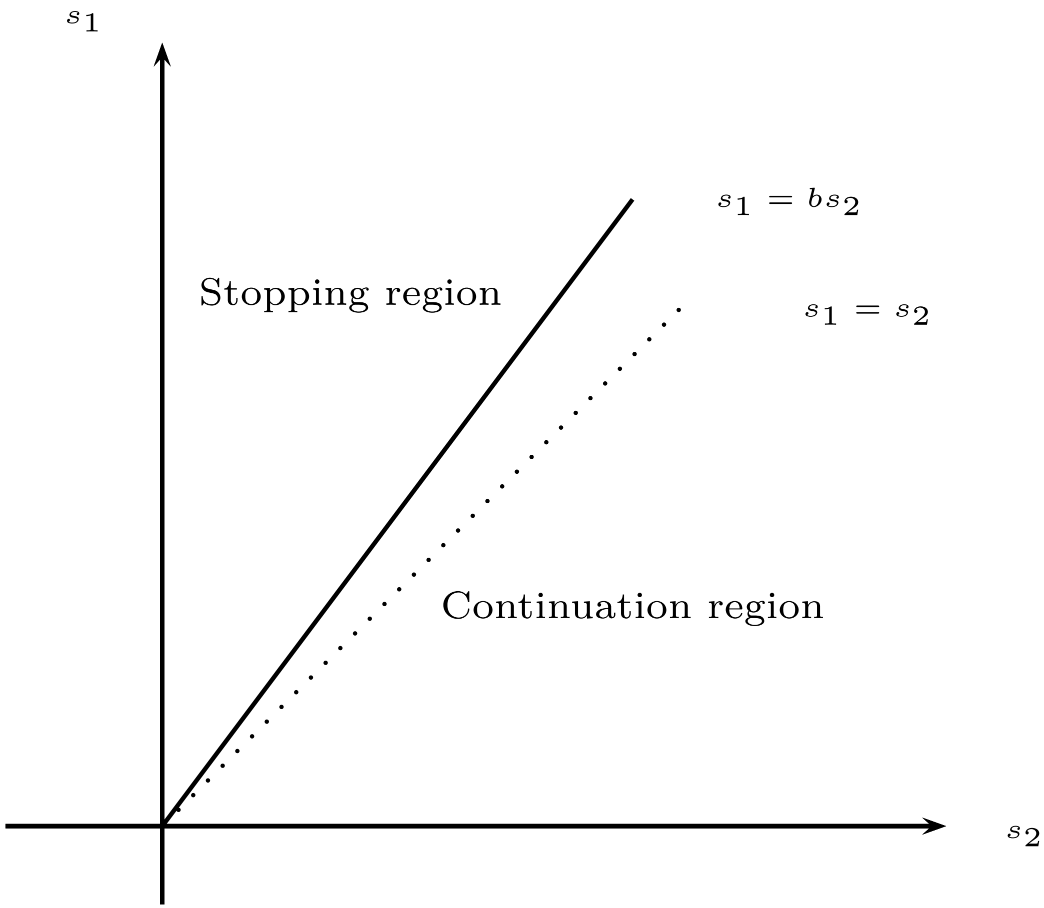

Figure 2: Continuation and stopping regions for the perpetual exchange option, at a given time t. The early exercise boundary is the line $s_1 = bs_2$.

- In contrast to the American exchange option under geometric Brownian motion (Broadie and Detemple (1997)) and the American exchange option under jump-diffusion (Cheang and Chiarella (2011)), the early exercise exercise boundary $s_1 = bs_2$ in Figure 2 is independent of time.
- In the finite time maturity case, the early exercise boundary is a straight line $s_1 = b_t s_2$ where $b_t > 1$ and $\lim_{t \to T} b_t = 1$ in Figure 1.

- In the case of single stock perpetual call and perpetual put options, Gerber and Shiu (1994, 1998) showed that optimal solutions exist for perpetual calls if the stock is skip-free upwards, whereas those of perpetual puts exist if the stock is skip-free downwards.
- Since the payoff of the exchange option takes the form $(S_{1,\tau} - S_{2,\tau})^+$, we impose the condition that stock S_1 is skip-free upwards and stock S_2 is skip-free downwards and it turns out that under these conditions, there is a close-form formula for the perpetual exchange option under jump-diffusion dynamics.

8 Exchange Options under SVJD

- Cheang and Garces (2020) extended the Cheang and Chiarella (2011) exchange option under jump-diffusion model to also include stochastic volatility for underlying stock prices.
- This is also an extension of the Cheang et al. (2013) call option on a single stock under the stochastic volatility jump-diffusion model to the scenario fo the exhange option where both stock prices exhibit stochastic volatility and jump-diffusion characteristcs.

- The dynamics for the stock prices in the Cheang and Garces (2020) model under an equivalent martingale measure $\mathbb{Q}$ for $i = 1, 2$ are

$$\frac{dS_{i,t}}{S_{i,t-}} = (r - \xi_i - \tilde{\lambda}_i \tilde{\kappa}_i)dt + \sqrt{v_{i,t}} d\widetilde{W}_{i,t} + [e^{Y_i} - 1]dN_{i,t}, \tag{23}$$

and

$$dv_{i,t} = \theta_i(\eta_i - v_{i,t})dt + \sigma_i \sqrt{v_{i,t}} d\widetilde{Z}_{i,t}, \tag{24}$$

where the $N_{i,t}$ are independent Poisson processes with rate $\tilde{\lambda}_i$, ξ_i is the instantaneous dividend rate, $\tilde{\kappa}_i = \mathbb{E}_{\mathbb{Q}}[e^{Y_i} - 1]$ and the correlation structure of the Brownian motion components $\widetilde{W}_{i.t}$ and $\widetilde{Z}_{i,t}$ are given by

	$\widetilde{W}_{1,t}$	$\widetilde{W}_{2,t}$	$\widetilde{Z}_{1,t}$	$\widetilde{Z}_{2,t}$
$\widetilde{W}_{1,t}$	1	ρ_W	ρ_{WZ_1}	0
$\widetilde{W}_{2,t}$	ρ_W	1	0	ρ_{WZ_2}
$\widetilde{Z}_{1,t}$	ρ_{WZ_1}	0	1	ρ_Z
$\widetilde{Z}_{2,t}$	0	ρ_{WZ_2}	ρ_Z	1

- In this model, the jump components are independent, the Brownian motion for the volatility of the ith stock is correlated with the Brownian motion for the diffusion part of the ith stock, the Brownian motion components for the diffusion parts of both stocks are correlated, and the Brownian motion components for the two volatility processes are also correlated.

- Some conditions are also required for the two variance processes in Equation (24) so that neither of the variances (volatilities) go to zero or explode to infinity.
- As with all option pricing problems where stochastic volatility is involved, there is also no close-form formula for the exchange option price under SVJD dynamics. However a close-form formula in terms of the Fourier transform of the exchange option price is obtained.
- If specific parameter values are given, it is possible to invert the Fourier transform numerically.

- Since there is stochastic volatility for both stocks, the exchange option prices for this model will also be functions of the variance processes of each stock as well as the individual stock prices, thus $C_t^E(S_{1,t}, S_{2,t}, v_{1,t}, v_{2,t})$ for the European exchange option price and $C_t^A(S_{1,t}, S_{2,t}, v_{1,t}, v_{2,t})$ for the American exchange option price.
- We are also able to show (as an inverted Fourier transform of the joint density) that the European exchange option price at time $t = 0$ has the decomposition

$$C_0^E(S_{1,0}, S_{2,0}, v_{1,0}, v_{2,0}) = S_{1,0}e^{-\xi_1 T}\mathbb{Q}_1\{A\} - S_{2,0}e^{-\xi_2 T}\mathbb{Q}_2\{A\}, \tag{25}$$

where A is the event $S_{1,T} > S_{2,T}$ (that the option is exercised at maturity), $\mathbb{Q}_1$ is the probability measure when all assets are priced in terms of units of $S_{1,t}e^{\xi_1 t}$, and $\mathbb{Q}_2$ is the probability measure when all assets are priced in terms of units of $S_{2,t}e^{\xi_2 t}$.

- In financial mathematics, we formally say that $\mathbb{Q}_1$ is the probability measure associated with $S_{1,t}e^{\xi_1 t}$ as the numéraire, and $\mathbb{Q}_2$ is the probability measure associated with $S_{2,t}e^{\xi_2 t}$ as the numéraire.

- It should be noted that for the model in Cheang and Chiarella (2011), a similar decomposition was obtained for the European exchange option

$$C_0^E(S_{1,0}, S_{2,0}) = S_{1,0}e^{-\xi_1 T}\mathbb{Q}_1\{A\} - S_{2,0}e^{-\xi_2 T}\mathbb{Q}_2\{A\}, \qquad (26)$$

where the close-form expression can be obtained directly without the need for Fourier transform methods.

- The American exchange option price for this model also has the decomposition

$$C_t^A(S_{1,t}, S_{2,t}, v_{1,t}, v_{2,t}) = C_t^E(S_{1,t}, S_{2,t}, v_{1,t}, v_{2,t}) + C_t^P(S_{1,t}, S_{2,t}, v_{1,t}, v_{2,t}), \tag{27}$$

  where $C_t^P(S_{1,t}, S_{2,t}, v_{1,t}, v_{2,t})$ is the early exercise premium term.

- There are also similar interpretations of the early exercise premium term $C_t^P(S_{1,t}, S_{2,t}, v_{1,t}, v_{2,t})$ (as in Cheang and Chiarella, 2011) when jumps in the stock prices happen to occur at the instant of time of exercise in this context.

- A slightly different model is considered in Garces and Cheang (2021).

- There the stock price dynamics are

$$\frac{dS_{i,t}}{S_{i,t-}} = (r - \xi_i - \tilde{\lambda}_i \tilde{\kappa}_i)dt + \sigma_i \sqrt{v_t} d\widetilde{W}_{i,t} + [e^{Y_i} - 1]dN_{i,t}, \qquad (28)$$

and

$$dv_t = \theta_i(\eta_i - v_t)dt + w\sqrt{v_t} d\widetilde{Z}_t, \qquad (29)$$

for $i = 1, 2$.

- This model is known as the proportional stochastic volatility and jump-diffusion (SVJD) model. While there is only one variance processes feeding into the diffusion component of each of the asset prices, the degree of influence the stochastic volatility process has on each stock price dynamics is governed by the proportionality coefficients σ_1 and σ_2 in Equation (28).

- In Garces and Cheang (2021), specific values are chosen for the parameters and the Method of Lines algorithm is used to computer the American exchange option price and the early exercise boundary estimates. The computed prices were also compared to the computed prices by other methods such as least squares Monte Carlo.

9 Conclusion

- We first gave an overview of modern option pricing theory based on a single stock/asset driven by a pure-diffusion model.
- We looked at the original Margrabe (1978) exchange option pricing model.
- We discussed the inadequacy of stock pricing models that are driven solely by pure-diffusion processes.
- We outlined further extensions to the Margrabe (1978) model by Cheang and Chiarella (2011), Cheang and Lian (2015), Cheang and Garces (2020), and Garces and Cheang (2021), where these extensions included American and Perpetual exchange options under jump-diffusion pricing models as well as American exchange options under SVJD pricing models.

- An understanding of the mathematics of exchange options under these extended models could possibly lead to a better understanding of spread options under these extended models.
- Other possible extensions for future work could include other multi-stock options driven by more general Lévy processes.

Terima Kasih!

Any Questions?

For the papers and comments, please email gerald.cheang@unisa.edu.au

https://www.researchgate.net/profile/Gerald-Cheang

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